

# Partition function of the eight-vertex model with domain wall boundary condition

Wen-Li Yang<sup>a,b</sup> and Yao-Zhong Zhang<sup>b</sup>

<sup>a</sup> Institute of Modern Physics, Northwest University, Xian 710069, P.R. China

<sup>b</sup> The University of Queensland, School of Mathematics and Physics, Brisbane, QLD 4072, Australia

## Abstract

We derive the recursive relations of the partition function for the eight-vertex model on an  $N \times N$  square lattice with domain wall boundary condition. Solving the recursive relations, we obtain the explicit expression of the domain wall partition function of the model. In the trigonometric/rational limit, our results recover the corresponding ones for the six-vertex model.

*PACS:* 75.10.Pq, 04.20.Jb, 05.50.+q

*Keywords:* Eight-vertex model; Partition function; Domain wall boundary condition.

# 1 Introduction

The domain wall boundary condition (DW) for the six vertex model on a finite square lattice was introduced by Korepin in [1], where some recursion relations of the partition function which fully determine the partition function were also derived. It was then found in [2, 3] that the partition function can be represented as a determinant. Such an explicit expression of the partition function has played an important role in constructing norms of Bethe states, correction functions [4, 5, 6] and thermodynamical properties of the six-vertex model [7, 8], and also in the Toda theories [9]. Moreover, it has been proven to be very useful in solving some pure mathematical problems, such as the problem of alternating sign matrices [10]. Recently, the partition functions with DW boundary condition have been obtained for the high-spin models [11] and the fermionic models [12, 13].

Among solvable models, elliptic ones stand out as a particularly important class due to the fact that most trigonometric and rational models can be obtained from them by certain limits. In this paper, we focus on the most fundamental elliptic model—the eight-vertex model [14, 15] whose trigonometric limit gives the six-vertex model. By means of the algebraic Bethe ansatz method we derive an explicit expression of the partition function for the eight-vertex model on an  $N \times N$  square lattice with the DW boundary condition. In the trigonometric limit, our results recover those obtained by Korepin et al in [1, 2, 3] for the six-vertex model.

The paper is organized as follows. In section 2, we introduce our notation and some basic ingredients. In section 3, after briefly reviewing the vertex-face correspondence, we introduce the four boundary states which specify the DW boundary condition of the eight-vertex model. In section 4, some properties of the partition function of the eight-vertex model with the DW boundary condition are obtained by using algebraic Bethe ansatz method. With help of these properties, we derive in section 5 the recursive relations of the partition function and obtain the explicit expression of the DW partition function by resolving the recursive relations. In section 6, we summarize our results and give some discussions. Some detailed technical proofs are given in Appendices A-B.

## 2 The Eight-vertex model

In this section, we define the DW boundary condition for the eight-vertex model on an  $N \times N$  square lattice [15].

### 2.1 The eight-vertex R-matrix

Let us fix  $\tau$  such that  $\text{Im}(\tau) > 0$  and a generic complex number  $\eta$ . Introduce the following elliptic functions

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (u, \tau) = \sum_{n=-\infty}^{\infty} \exp \left\{ i\pi \left[ (n+a)^2 \tau + 2(n+a)(u+b) \right] \right\}, \quad (2.1)$$

$$\theta^{(j)}(u) = \theta \begin{bmatrix} \frac{1}{2} - \frac{j}{2} \\ \frac{1}{2} \end{bmatrix} (u, 2\tau), \quad j = 1, 2, \quad (2.2)$$

$$\sigma(u) = \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (u, \tau), \quad \sigma'(u) = \frac{\partial}{\partial u} \{\sigma(u)\}. \quad (2.3)$$

The  $\sigma$ -function<sup>1</sup> satisfies the so-called Riemann identity:

$$\begin{aligned} & \sigma(u+x)\sigma(u-x)\sigma(v+y)\sigma(v-y) - \sigma(u+y)\sigma(u-y)\sigma(v+x)\sigma(v-x) \\ &= \sigma(u+v)\sigma(u-v)\sigma(x+y)\sigma(x-y), \end{aligned} \quad (2.4)$$

which will be useful in the following.

Let  $V$  be a two-dimensional vector space  $\mathbb{C}^2$  and  $\{\epsilon_i | i = 1, 2\}$  be the orthonormal basis of  $V$  such that  $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$ . The well-known eight-vertex model R-matrix  $R(u) \in \text{End}(V \otimes V)$  is given by

$$R(u) = \begin{pmatrix} a(u) & & d(u) & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{pmatrix}. \quad (2.5)$$

The non-vanishing matrix elements are [15]

$$\begin{aligned} a(u) &= \frac{\theta^{(1)}(u) \theta^{(0)}(u+\eta) \sigma(\eta)}{\theta^{(1)}(0) \theta^{(0)}(\eta) \sigma(u+\eta)}, & b(u) &= \frac{\theta^{(0)}(u) \theta^{(1)}(u+\eta) \sigma(\eta)}{\theta^{(1)}(0) \theta^{(0)}(\eta) \sigma(u+\eta)}, \\ c(u) &= \frac{\theta^{(1)}(u) \theta^{(0)}(u+\eta) \sigma(\eta)}{\theta^{(1)}(0) \theta^{(1)}(\eta) \sigma(u+\eta)}, & d(u) &= \frac{\theta^{(0)}(u) \theta^{(0)}(u+\eta) \sigma(\eta)}{\theta^{(1)}(0) \theta^{(1)}(\eta) \sigma(u+\eta)}. \end{aligned} \quad (2.6)$$

---

<sup>1</sup>Our  $\sigma$ -function is the  $\vartheta$ -function  $\vartheta_1(u)$  [16]. It has the following relation with the *Weierstrassian*  $\sigma$ -function  $\sigma_w(u)$ :  $\sigma_w(u) \propto e^{\eta_1 u^2} \sigma(u)$  with  $\eta_1 = \pi^2 \left( \frac{1}{6} - 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} \right)$  and  $q = e^{i\tau}$ .

Here  $u$  is the spectral parameter and  $\eta$  is the so-called crossing parameter. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{1,2}(u_1 - u_2)R_{1,3}(u_1 - u_3)R_{2,3}(u_2 - u_3) = R_{2,3}(u_2 - u_3)R_{1,3}(u_1 - u_3)R_{1,2}(u_1 - u_2), \quad (2.7)$$

and the properties,

$$Z_2\text{-symmetry : } \sigma_1^i \sigma_2^i R_{1,2}(u) = R_{1,2}(u) \sigma_1^i \sigma_2^i, \quad \text{for } i = x, y, z, \quad (2.8)$$

$$\text{Initial condition : } R_{1,2}(0) = P_{12}, \quad (2.9)$$

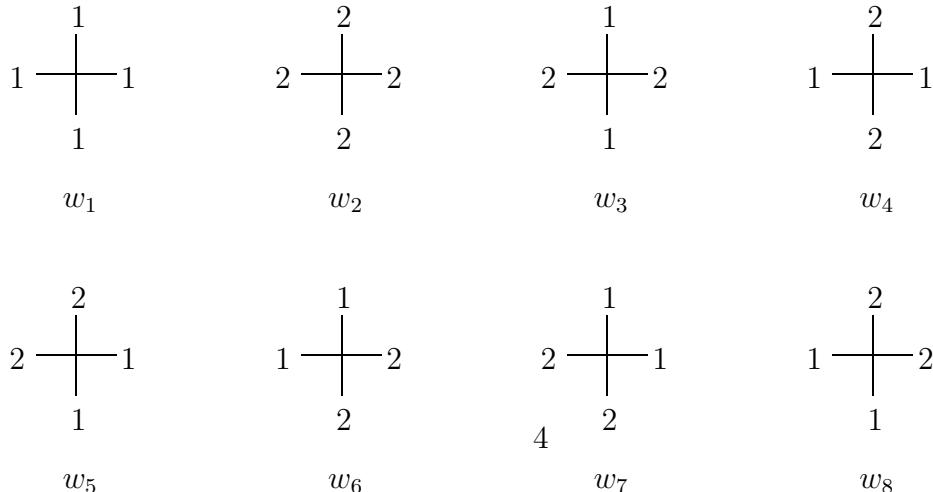
Here  $\sigma^x, \sigma^y, \sigma^z$  are the Pauli matrices and  $P_{12}$  is the usual permutation operator. Throughout this paper we adopt the standard notations: for any matrix  $A \in \text{End}(V)$ ,  $A_j$  is an embedding operator in the tensor space  $V \otimes V \otimes \dots$ , which acts as  $A$  on the  $j$ -th space and as identity on the other factor spaces;  $R_{i,j}(u)$  is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the  $i$ -th and  $j$ -th ones.

## 2.2 The model

The partition function of a statistical model on a two-dimensional lattice is defined by the following:

$$Z = \sum \exp\left\{-\frac{E}{kT}\right\},$$

where  $E$  is the energy of the system,  $k$  is the Boltzmann constant,  $T$  is the temperature of the system, and the summation is taken over all possible configurations under the particular boundary condition such as the DW boundary condition. The model we consider here has eight allowed local vertex configurations



**Figure 1.** Vertex configurations and their associated Boltzmann weights.

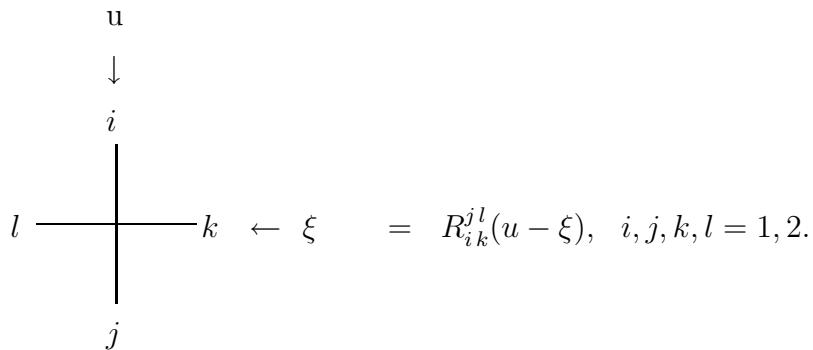
where 1 and 2 respectively denote the spin up and down states. Each of these eight configurations is assigned a statistical weight (or Boltzmann weight)  $w_i$ . Then the partition function can be rewritten as

$$Z = \sum w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} w_5^{n_5} w_6^{n_6} w_7^{n_7} w_8^{n_8},$$

where the summation is over all possible vertex configurations with  $n_i$  being the number of the vertices of type  $i$ . If the local Boltzmann weights have  $Z_2$ -symmetry, i.e.,

$$a \equiv w_1 = w_2, \quad b \equiv w_3 = w_4, \quad c \equiv w_5 = w_6, \quad d \equiv w_7 = w_8, \quad (2.10)$$

and the variables  $a, b, c, d$  satisfy a function relation, or equivalently, the local Boltzmann weights  $\{w_i\}$  can be parameterized by the matrix elements of the eight-vertex R-matrix  $R$  (2.5)-(2.6) as in figure 2,

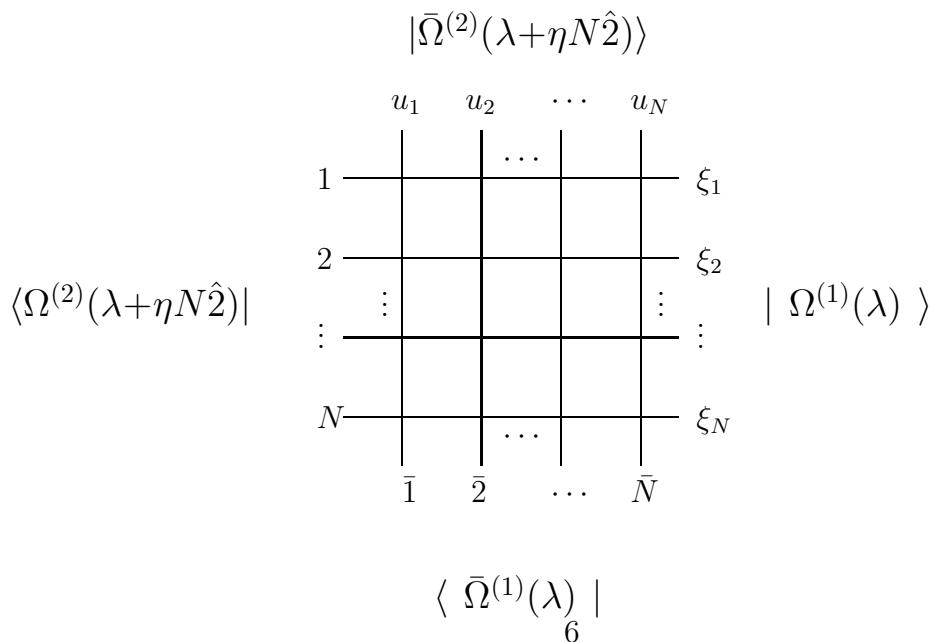


**Figure 2.** The Boltzmann weights and elements of the eight-vertex R-matrix.

then the corresponding model is called the eight-vertex model which can be exactly solved [15]. Therefore the partition function of the eight-vertex model is given by

$$Z = \sum a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6} d^{n_7+n_8}.$$

In order to parameterize the local Boltzmann weights in terms of the elements of the R-matrix, one needs to assign spectral parameters  $u$  and  $\xi$  respectively to the vertical line and horizontal line of each vertex of the lattice, as shown in figure 2. In an inhomogeneous model, the statistical weights are site-dependent. Hence two sets of spectral parameters  $\{u_\alpha\}$  and  $\{\xi_i\}$  are needed, see figure 3. The horizontal lines are enumerated by indices  $1, \dots, N$  with spectral parameters  $\{\xi_i\}$ , while the vertical lines are enumerated by indices  $\bar{1}, \dots, \bar{N}$  with spectral parameters  $\{u_\alpha\}$ . The DW boundary condition is specified by four boundary states  $|\Omega^{(1)}(\lambda)\rangle$ ,  $|\bar{\Omega}^{(1)}(\lambda)\rangle$ ,  $\langle\Omega^{(2)}(\lambda + \eta N\hat{2})|$  and  $\langle\bar{\Omega}^{(2)}(\lambda + \eta N\hat{2})|$  (the definitions of the boundary states will be given later in section 3, see (3.12)-(3.15) below). These four states correspond to the particular choices of spin states on the four boundaries of the lattice. In contrast to the six-vertex case [4], our boundary states not only depend on the spectral parameters ( $|\Omega^{(1)}(\lambda)\rangle$  and  $\langle\Omega^{(2)}(\lambda + \eta N\hat{2})|$  depend on  $\{\xi_i\}$ , while  $|\bar{\Omega}^{(1)}(\lambda)\rangle$  and  $\langle\bar{\Omega}^{(2)}(\lambda + \eta N\hat{2})|$  depend on  $\{u_\alpha\}$ ) but also on two continuous parameters  $\lambda_1$  and  $\lambda_2$  (it is convenient to introduce a vector  $\lambda \in V$  associated with these two parameters  $\{\lambda_i\}$ :  $\lambda = \sum_{i=1}^2 \lambda_i \epsilon_i$ ). However, in the trigonometric limit (i.e., setting  $\lambda_2 = \frac{\tau}{2}$  and then taking  $\tau \rightarrow +i\infty$ ), the corresponding boundary states  $|\Omega^{(1)}(\lambda)\rangle$  and  $\langle\bar{\Omega}^{(1)}(\lambda)|$  (or  $|\bar{\Omega}^{(2)}(\lambda + \eta N\hat{2})\rangle$  and  $\langle\Omega^{(2)}(\lambda + \eta N\hat{2})|$ ) become the state of all spin up and its dual (or the state of all spin down and its dual) up to some over-all scalar factors. Therefore the partition function in the limit reduces to that of the six-vertex model [1, 2, 3]. In this sense, we call the partition function corresponding to the boundary condition given in figure 3 the DW partition function of the eight-vertex model.



**Figure 3.** The eight-vertex model with DW boundary condition.

Now the partition function of the eight-vertex model with DW boundary condition is a function of  $2N+2$  variables  $\{u_\alpha\}$ ,  $\{\xi_i\}$ ,  $\lambda_1$  and  $\lambda_2$ , which will be denoted by  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$ . Due to the fact that the local Boltzmann weights of each vertex of the lattice are given by the matrix elements of the eight-vertex R-matrix (see figure 2), the partition function can be expressed in terms of the product of the R-matrices and the four boundary states

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) = & \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | \langle \bar{\Omega}^{(1)}(\lambda) | R_{\bar{1},N}(u_1 - \xi_N) \dots R_{\bar{1},1}(u_1 - \xi_1) \dots \\ & \times R_{\bar{N},N}(u_N - \xi_N) \dots R_{\bar{N},1}(u_N - \xi_1) | \bar{\Omega}^{(2)}(\lambda + \eta N \hat{2}) \rangle | \Omega^{(1)}(\lambda) \rangle. \end{aligned} \quad (2.11)$$

The aim of this paper is to obtain an explicit expression for  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$ .

One can rearrange the product of the R-matrices in (2.11) in terms of a product of the row-to-row monodromy matrices, namely,

$$Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) = \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | \langle \bar{\Omega}^{(1)}(\lambda) | T_{\bar{1}}(u_1) \dots T_{\bar{N}}(u_N) | \bar{\Omega}^{(2)}(\lambda + \eta N \hat{2}) \rangle | \Omega^{(1)}(\lambda) \rangle, \quad (2.12)$$

where the monodromy matrix  $T_{\bar{i}}(u)$  is given by

$$T_{\bar{i}}(u) \equiv T_{\bar{i}}(u; \xi_1, \dots, \xi_N) = R_{\bar{i},N}(u - \xi_N) \dots R_{\bar{i},1}(u - \xi_1). \quad (2.13)$$

The QYBE (2.7) of the R-matrix gives rise to the so-called “RLL” relation satisfied by the monodromy matrix  $T_{\bar{i}}(u)$ ,

$$R_{\bar{i},\bar{j}}(u_i - u_j) T_{\bar{i}}(u_i) T_{\bar{j}}(u_j) = T_{\bar{j}}(u_j) T_{\bar{i}}(u_i) R_{\bar{i},\bar{j}}(u_i - u_j). \quad (2.14)$$

### 3 The boundary states

#### 3.1 The Vertex-Face correspondence

Let us briefly review the face-type R-matrix associated with the eight-vertex model. From the orthonormal basis  $\{\epsilon_i\}$  of  $V$ , we define

$$\hat{i} = \epsilon_i - \bar{\epsilon}, \quad \bar{\epsilon} = \frac{1}{2} \sum_{k=1}^2 \epsilon_k, \quad i = 1, 2, \quad \text{then} \quad \sum_{k=1}^2 \hat{k} = 0. \quad (3.1)$$

For a generic  $m \in V$ , define

$$m_i = \langle m, \epsilon_i \rangle, \quad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j \rangle, \quad i, j = 1, 2. \quad (3.2)$$

Let  $\overline{R}(u; m) \in \text{End}(V \otimes V)$  be the R-matrix of the eight-vertex SOS model [15] given by

$$\overline{R}(u; m) = \sum_{i=1}^2 \overline{R}_{ii}^{ii}(u; m) E_{ii} \otimes E_{ii} + \sum_{i \neq j}^2 \left\{ \overline{R}_{ij}^{ij}(u; m) E_{ii} \otimes E_{jj} + \overline{R}_{ij}^{ji}(u; m) E_{ji} \otimes E_{ij} \right\}, \quad (3.3)$$

where  $E_{ij}$  is the matrix with elements  $(E_{ij})_k^l = \delta_{jk} \delta_{il}$ . The coefficient functions are

$$\overline{R}_{ii}^{ii}(u; m) = 1, \quad \overline{R}_{ij}^{ij}(u; m) = \frac{\sigma(u)\sigma(m_{ij} - \eta)}{\sigma(u + \eta)\sigma(m_{ij})}, \quad i \neq j, \quad (3.4)$$

$$\overline{R}_{ij}^{ji}(u; m) = \frac{\sigma(\eta)\sigma(u + m_{ij})}{\sigma(u + \eta)\sigma(m_{ij})}, \quad i \neq j, \quad (3.5)$$

and  $m_{ij}$  is defined in (3.2). The R-matrix  $\overline{R}$  satisfies the dynamical (modified) quantum Yang-Baxter equation (or star-triangle equation) [15].

Let us introduce two intertwiners which are 2-component column-vectors  $\phi_{m, m-\eta\hat{j}}(u)$  labelled by  $\hat{j} = \hat{1}, \hat{2}$ . The  $k$ -th element of  $\phi_{m, m-\eta\hat{j}}(u)$  is given by

$$\phi_{m, m-\eta\hat{j}}^{(k)}(u) = \theta^{(k)}(u + 2m_j). \quad (3.6)$$

Explicitly,

$$\phi_{m, m-\eta\hat{1}}(u) = \begin{pmatrix} \theta^{(1)}(u + 2m_1) \\ \theta^{(2)}(u + 2m_1) \end{pmatrix}, \quad \phi_{m, m-\eta\hat{2}}(u) = \begin{pmatrix} \theta^{(1)}(u + 2m_2) \\ \theta^{(2)}(u + 2m_2) \end{pmatrix}.$$

It is easy to check that these two intertwiner vectors  $\phi_{m, m-\eta\hat{j}}(u)$  are linearly *independent* for a generic  $m \in V$ .

Using the intertwiner vectors, one can derive the following vertex-face correspondence relation [15, 17]

$$\begin{aligned} & R_{1,2}(u_1 - u_2) \phi_{m, m-\eta\hat{j}}^1(u_1) \phi_{m-\eta\hat{l}, m-\eta(\hat{i}+\hat{j})}^2(u_2) \\ &= \sum_{k,l} \overline{R}(u_1 - u_2; m)_{ij}^{kl} \phi_{m-\eta\hat{l}, m-\eta(\hat{l}+\hat{k})}^1(u_1) \phi_{m, m-\eta\hat{l}}^2(u_2). \end{aligned} \quad (3.7)$$

Hereafter we adopt the convention:  $\phi^1 = \phi \otimes \text{id} \otimes \dots$ ,  $\phi^2 = \text{id} \otimes \phi \otimes \text{id} \otimes \dots$ , etc. The QYBE (2.7) of the vertex-type R-matrix  $R(u)$  is equivalent to the dynamical Yang-Baxter

equation of the SOS R-matrix  $\bar{R}(u; m)$ . For a generic  $m$ , we can introduce two row-vector intertwiners  $\tilde{\phi}$  satisfying the conditions,

$$\tilde{\phi}_{m+\eta\hat{\mu},m}(u) \phi_{m+\eta\hat{\nu},m}(u) = \delta_{\mu\nu}, \quad \mu, \nu = 1, 2, \quad (3.8)$$

from which one derives the relation,

$$\sum_{\mu=1}^2 \phi_{m+\eta\hat{\mu},m}(u) \tilde{\phi}_{m+\eta\hat{\mu},m}(u) = \text{id}. \quad (3.9)$$

With the help of (3.6)-(3.9), we obtain the following relations from the vertex-face correspondence relation (3.7):

$$\begin{aligned} & \tilde{\phi}_{m+\eta\hat{k},m}^1(u_1) R_{1,2}(u_1 - u_2) \phi_{m+\eta\hat{j},m}^2(u_2) \\ &= \sum_{i,l} \bar{R}(u_1 - u_2; m)_{ij}^{kl} \tilde{\phi}_{m+\eta(\hat{i}+\hat{j}),m+\eta\hat{j}}^1(u_1) \phi_{m+\eta(\hat{k}+\hat{l}),m+\eta\hat{k}}^2(u_2), \end{aligned} \quad (3.10)$$

$$\begin{aligned} & \tilde{\phi}_{m+\eta\hat{k},m}^1(u_1) \tilde{\phi}_{m+\eta(\hat{k}+\hat{l}),m+\eta\hat{k}}^2(u_2) R_{1,2}(u_1 - u_2) \\ &= \sum_{i,j} \bar{R}(u_1 - u_2; m)_{ij}^{kl} \tilde{\phi}_{m+\eta(\hat{i}+\hat{j}),m+\eta\hat{j}}^1(u_1) \tilde{\phi}_{m+\eta\hat{j},m}^2(u_2). \end{aligned} \quad (3.11)$$

The intertwiners  $\phi$ ,  $\tilde{\phi}$  and the associated vertex-face correspondence relations will play an important role in determining the very properties of the partition function  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  that enable us in section 4 to fully determine its explicit expression.

## 3.2 The boundary states

Now we are in the position to construct the boundary states which have been used in section 2 to specify the DW boundary condition of the eight-vertex model, see figure 3.

For any vector  $m \in V$ , we introduce four states<sup>2</sup> which live in the two  $N$ -tensor spaces of  $V$  (one is indexed by  $1, \dots, N$  and the other is indexed by  $\bar{1}, \dots, \bar{N}$ ) or their dual spaces as follows:

$$|\Omega^{(i)}(m)\rangle = \phi_{m,m-\eta\hat{i}}^1(\xi_1) \phi_{m-\eta\hat{i},m-2\eta\hat{i}}^2(\xi_2) \dots \phi_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}^N(\xi_N), \quad i = 1, 2, \quad (3.12)$$

$$|\bar{\Omega}^{(i)}(m)\rangle = \phi_{m,m-\eta\hat{i}}^{\bar{1}}(u_1) \phi_{m-\eta\hat{i},m-2\eta\hat{i}}^{\bar{2}}(u_2) \dots \phi_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}^{\bar{N}}(u_N), \quad i = 1, 2, \quad (3.13)$$

---

<sup>2</sup>Among them,  $|\Omega^{(i)}(m)\rangle$ , with special choices of  $m$ , are the complete reference states of the open XYZ spin chain [18], and have played an important role in constructing the extra center elements of the elliptic algebra at roots of unity [19, 20]

$$\langle \Omega^{(i)}(m) | = \tilde{\phi}_{m,m-\eta\hat{i}}^1(\xi_1) \tilde{\phi}_{m-\eta\hat{i},m-2\eta\hat{i}}^2(\xi_2) \dots \tilde{\phi}_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}^N(\xi_N), \quad i = 1, 2, \quad (3.14)$$

$$\langle \bar{\Omega}^{(i)}(m) | = \tilde{\phi}_{m,m-\eta\hat{i}}^{\bar{1}}(u_1) \tilde{\phi}_{m-\eta\hat{i},m-2\eta\hat{i}}^{\bar{2}}(u_2) \dots \tilde{\phi}_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}^{\bar{N}}(u_N), \quad i = 1, 2. \quad (3.15)$$

The boundary states which have been used to define the DW boundary condition in section 2 can be obtained through the above states by special choices of  $m$  and  $i$  (for example,  $m$  is specified to  $\lambda$  or  $\lambda + \eta N\hat{2}$ ). Then the DW partition function  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  given by (2.11) becomes

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) = & \tilde{\phi}_{\lambda+\eta N\hat{2}, \lambda+\eta(N-1)\hat{2}}^1(\xi_1) \dots \tilde{\phi}_{\lambda+\eta\hat{2}, \lambda}^N(\xi_N) \tilde{\phi}_{\lambda, \lambda-\eta\hat{1}}^{\bar{1}}(u_1) \dots \tilde{\phi}_{\lambda-\eta(N-1)\hat{1}, \lambda-\eta N\hat{1}}^{\bar{N}}(u_N) \\ & \times R_{\bar{1}, N}(u_1 - \xi_N) \dots R_{\bar{1}, 1}(u_1 - \xi_1) \dots R_{\bar{N}, N}(u_N - \xi_N) \dots R_{\bar{N}, 1}(u_N - \xi_1) \\ & \times \phi_{\lambda+\eta N\hat{2}, \lambda+\eta(N-1)\hat{2}}^{\bar{1}}(u_1) \dots \phi_{\lambda+\eta\hat{2}, \lambda}^{\bar{N}}(u_N) \phi_{\lambda, \lambda-\eta\hat{1}}^1(\xi_1) \dots \phi_{\lambda-\eta(N-1)\hat{1}, \lambda-\eta N\hat{1}}^N(\xi_N). \end{aligned} \quad (3.16)$$

## 4 The properties of the partition function

In this section we will derive certain properties of the DW partition function  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  which enable us to determine its explicit expression.

For the case of  $N = 1$ , the corresponding partition function (3.16) becomes

$$\begin{aligned} Z_1(u_1; \xi_1; \lambda) &= \tilde{\phi}_{\lambda+\eta\hat{2}, \lambda}^1(\xi_1) \tilde{\phi}_{\lambda, \lambda-\eta\hat{1}}^{\bar{1}}(u_1) R_{\bar{1}, 1}(u_1 - \xi_1) \phi_{\lambda+\eta\hat{2}, \lambda}^{\bar{1}}(u_1) \phi_{\lambda, \lambda-\eta\hat{1}}^1(\xi_1) \\ &\stackrel{(3.7)}{=} \sum_{k, l=1}^2 \bar{R}_{21}^{k l}(u_1 - \xi_1; \lambda + \eta\hat{2}) \left( \tilde{\phi}_{\lambda+\eta\hat{2}, \lambda}^1(\xi_1) \phi_{\lambda+\eta\hat{2}, \lambda+\eta(\hat{2}-\hat{l})}^1(\xi_1) \right) \\ &\quad \times \left( \tilde{\phi}_{\lambda, \lambda-\eta\hat{1}}^{\bar{1}}(u_1) \phi_{\lambda+\eta(\hat{k}-\hat{1}), \lambda-\eta\hat{1}}^{\bar{1}}(u_1) \right) \\ &\stackrel{(3.8)}{=} \bar{R}_{21}^{1 2}(u_1 - \xi_1; \lambda + \eta\hat{2}). \end{aligned}$$

Thus, we have the first property of the partition function:

$$Z_1(u_1; \xi_1; \lambda) = \frac{\sigma(\eta) \sigma(u_1 - \xi_1 + \lambda_{21} + \eta)}{\sigma(u_1 - \xi_1 + \eta) \sigma(\lambda_{21} + \eta)}, \quad (4.1)$$

where  $\lambda_{21}$  is defined by (3.2). Using the fundamental exchange relation (2.14) and the vertex-face relations (3.7), (3.10) and (3.11), we derive the second property of the partition function:

$$Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) \text{ is a symmetric function of } \{u_\alpha\} \text{ and } \{\xi_i\} \text{ separately.} \quad (4.2)$$

The proof of the above property is relegated to Appendix A.

In addition to the Riemann identity (2.4), the  $\sigma$ -function enjoys the following quasi-periodic properties:

$$\sigma(u+1) = -\sigma(u), \quad \sigma(u+\tau) = -e^{-2i\pi(u+\frac{\tau}{2})}\sigma(u), \quad (4.3)$$

which are useful in deriving the quasi-periodicity of the partition function. The expansions of the boundary states in terms of the intertwiner vectors (3.12)-(3.15) and the partition function in terms of the monodromy matrices (2.12) allow us to rewrite  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  as

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) = & \tilde{\phi}_{\lambda+\eta\hat{2}, \lambda+\eta(N-1)\hat{2}}^1(\xi_1) \dots \tilde{\phi}_{\lambda+\eta\hat{2}, \lambda}^N(\xi_N) \tilde{\phi}_{\lambda, \lambda-\eta\hat{1}}^{\bar{1}}(u_1) T_{\bar{1}}(u_1) \phi_{\lambda+\eta\hat{2}, \lambda+\eta(N-1)\hat{2}}^{\bar{1}}(u_1) \dots \\ & \times \tilde{\phi}_{\lambda-\eta(N-1)\hat{1}, \lambda-\eta\hat{1}}^{\bar{N}}(u_N) T_{\bar{N}}(u_N) \phi_{\lambda+\eta\hat{2}, \lambda}^{\bar{N}}(u_N) \phi_{\lambda, \lambda-\eta\hat{1}}^1(\xi_1) \dots \phi_{\lambda-\eta(N-1)\hat{1}, \lambda-\eta\hat{1}}^N(\xi_N). \end{aligned}$$

The dependence of  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  on the argument  $u_N$  only comes from the last term corresponding to the second line of the above equation. Let us denote the term by  $A(u_N)$ , namely,

$$\begin{aligned} A(u_N) &= \tilde{\phi}_{\lambda-\eta(N-1)\hat{1}, \lambda-\eta\hat{N}}^{\bar{N}}(u_N) T_{\bar{N}}(u_N) \phi_{\lambda+\eta\hat{2}, \lambda}^{\bar{N}}(u_N) \phi_{\lambda, \lambda-\eta\hat{1}}^1(\xi_1) \dots \phi_{\lambda-\eta(N-1)\hat{1}, \lambda-\eta\hat{N}}^N(\xi_N) \\ &= \tilde{\phi}_{\lambda-\eta(N-1)\hat{1}, \lambda-\eta\hat{N}}^{\bar{N}}(u_N) R_{\bar{N}, N}(u_N - \xi_N) \phi_{\lambda-\eta(N-1)\hat{1}, \lambda-\eta\hat{N}}^N(\xi_N) \dots \\ (3.10) \quad &\stackrel{=}{=} \phi_{\lambda-\eta(N-2)\hat{1}, \lambda-\eta(N-1)\hat{1}}^N(\xi_N) \tilde{\phi}_{\lambda-\eta(N-2)\hat{1}, \lambda-\eta(N-1)\hat{1}}^{\bar{N}}(u_N) R_{\bar{N}, N-1}(u_N - \xi_{N-1}) \dots \\ &\quad + \overline{R}(u_N - \xi_N; \lambda - \eta N \hat{1})_{21}^{12} \phi_{\lambda-\eta(N-2)\hat{1}+\eta\hat{2}, \lambda-\eta(N-1)\hat{1}}^N(\xi_N) \tilde{\phi}_{\lambda-\eta(N-2)\hat{1}+\eta\hat{2}, \lambda-\eta(N-1)\hat{1}}^{\bar{N}}(u_N) \dots \\ &= \\ &\vdots \\ &= \phi_{\lambda-\eta(N-2)\hat{1}, \lambda-\eta(N-1)\hat{1}}^N(\xi_N) \tilde{\phi}_{\lambda-\eta(N-2)\hat{1}, \lambda-\eta(N-1)\hat{1}}^{\bar{N}}(u_N) R_{\bar{N}, N-1}(u_N - \xi_{N-1}) \dots \\ &\quad + \overline{R}(u_N - \xi_N; \lambda - \eta N \hat{1})_{21}^{12} \prod_{l=1}^{N-1} \overline{R}(u_N - \xi_l; \lambda - \eta l \hat{1})_{21}^{21} \phi_{\lambda+\eta\hat{2}, \lambda+\eta\hat{2}-\eta\hat{1}}^1(\xi_1) \dots \end{aligned}$$

It can be shown by induction that  $A(u_N)$  satisfies the following quasi-periodicity:

$$A(u_N + 1) = A(u_N), \quad A(u_N + \tau) = e^{-2i\pi(\lambda_{21})} A(u_N).$$

Since  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  is a symmetric function of  $\{u_\alpha\}$ , we conclude that  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  has the following quasi-periodic properties

$$Z_N(u_1, \dots, u_l + 1, u_{l+1}, \dots; \{\xi_i\}; \lambda) = Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda), \quad l = 1, \dots, N, \quad (4.4)$$

$$Z_N(u_1, \dots, u_l + \tau, u_{l+1}, \dots; \{\xi_i\}; \lambda) = e^{-2i\pi(\lambda_{21})} Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda), \quad l = 1, \dots, N. \quad (4.5)$$

Using the expressions (3.4)-(3.5) of the matrix elements of the R-matrix  $\bar{R}$  and the vertex-face correspondences (3.7) and (3.10), we find the analytic property of the partition function:

$Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  is an analytic function of  $u_l$  with simple poles  $\{\xi_i - \eta | i = 1, \dots, N\}$  inside the fundamental (upright) rectangle [15] generated by 1 and  $\tau$ . (4.6)

Direct calculation (for details see Appendix B) shows that at each simple pole  $\xi_i - \eta$  the corresponding residue is

$$\text{Res}_{u_l=\xi_i-\eta} (Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)) = \frac{\sigma(\eta)\sigma(\lambda_{21})}{\sigma'(0)\sigma(\lambda_{21} + \eta)} \prod_{\alpha \neq l} \frac{\sigma(u_\alpha - \xi_i)}{\sigma(u_\alpha - \xi_i + \eta)} \prod_{j \neq i} \frac{\sigma(\xi_j - \xi_i + \eta)}{\sigma(\xi_j - \xi_i)} \times Z_{N-1}(\{u_\alpha\}_{\alpha \neq l}; \{\xi_j\}_{j \neq i}; \lambda + \eta \hat{2}), \quad l, i = 1, \dots, N. \quad (4.7)$$

Similarly, using the initial condition (2.9) of the R-matrix  $R$  and the vertex-face correspondences (3.7) and (3.10), one can also show that the partition function  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  satisfies the following relations

$$Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda) |_{u_l=\xi_i} = Z_{N-1}(\{u_\alpha\}_{\alpha \neq l}; \{\xi_j\}_{j \neq i}; \lambda), \quad l, i = 1, \dots, N. \quad (4.8)$$

So remarks are in order. The properties (4.1), (4.2), (4.4), (4.5), (4.6) and (4.7) uniquely determine the partition function. On the other hand, the properties (4.1), (4.2), (4.4), (4.5), (4.6) and (4.8) also fully fix the partition function. They yield the recursive relations (5.4) and (5.5) (see below) respectively.

## 5 The partition function

In this section, we will derive two recursive relations from the properties of the partition function obtained in the previous section. Each of the recursive relations together with (4.1) uniquely determines the partition function.

### 5.1 The recursive relation

We now concentrate on the  $u_N$ -dependence of the partition function  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$ . From (4.4) and (4.5), we have

$$\begin{aligned} Z_N(u_1, \dots, u_{N-1}, u_N + 1; \{\xi_j\}; \lambda) &= Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda), \\ Z_N(u_1, \dots, u_{N-1}, u_N + \tau; \{\xi_j\}; \lambda) &= e^{-2i\pi(\lambda_{21})} Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda). \end{aligned} \quad (5.1)$$

The analytic properties (4.6) and (4.7) imply that

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda) &= \sum_{i=1}^N \left\{ \frac{\sigma(\eta)\sigma(u_N - \xi_i + a_i)}{\sigma(u_N - \xi_i + \eta)\sigma(b_i)} \prod_{j \neq i} \frac{\sigma(\xi_j - \xi_i + \eta)}{\sigma(\xi_j - \xi_i)} \prod_{l \neq N} \frac{\sigma(u_l - \xi_i)}{\sigma(u_l - \xi_i + \eta)} \right. \\ &\quad \left. \times Z_{N-1}(\{u_\alpha\}_{\alpha \neq N}; \{\xi_j\}_{j \neq i}; \lambda + \eta \hat{2}) \right\} + \Delta, \end{aligned}$$

where  $\{a_i\}$ ,  $\{b_i\}$  and  $\Delta$  are some constants with respect to  $u_N$ , and  $a_i$  and  $b_i$  satisfy the constraints

$$\frac{\sigma(a_i - \eta)}{\sigma(b_i)} = \frac{\sigma(\lambda_{21})}{\sigma(\lambda_{21} + \eta)}, \quad i = 1, \dots, N. \quad (5.2)$$

The quasi-periodic condition (5.1) leads to

$$\begin{cases} a_i = \lambda_{21} + \eta & i = 1, \dots, N, \\ \Delta = 0. \end{cases} \quad (5.3)$$

The constraint (5.2) then yields that  $b_i = a_i = \lambda_{21} + \eta$ . Thus the partition function  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$  satisfies the following recursive relation

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda) &= \sum_{i=1}^N \left\{ \frac{\sigma(\eta)\sigma(u_N - \xi_i + \lambda_{21} + \eta)}{\sigma(u_N - \xi_i + \eta)\sigma(\lambda_{21} + \eta)} \prod_{j \neq i} \frac{\sigma(\xi_j - \xi_i + \eta)}{\sigma(\xi_j - \xi_i)} \prod_{l \neq N} \frac{\sigma(u_l - \xi_i)}{\sigma(u_l - \xi_i + \eta)} \right. \\ &\quad \left. \times Z_{N-1}(\{u_\alpha\}_{\alpha \neq N}; \{\xi_j\}_{j \neq i}; \lambda + \eta \hat{2}) \right\}. \end{aligned} \quad (5.4)$$

On the other hand, the quasi-periodicity (5.1) of the partition function, the fact that the partition function only has simple poles at  $\{\xi_i - \eta\}$  and the relation (4.8) imply that the partition function has the following expansion

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda) &= \sum_{i=1}^N \left\{ \frac{\sigma(\eta)\sigma(u_N - \xi_i + a'_i)}{\sigma(u_N - \xi_i + \eta)\sigma(a'_i)} \prod_{j \neq i} \frac{\sigma(u_N - \xi_j)\sigma(\xi_i - \xi_j + \eta)}{\sigma(u_N - \xi_j + \eta)\sigma(\xi_i - \xi_j)} \right. \\ &\quad \left. \times Z_{N-1}(\{u_\alpha\}_{\alpha \neq N}; \{\xi_j\}_{j \neq i}; \lambda) \right\}, \end{aligned}$$

where  $\{a_i\}$  are some constants with respect to  $u_N$ . The quasi-periodicity (5.1) further requires  $a'_i = \lambda_{21} + N\eta$ . Namely, the partition function  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$  satisfies the following recursive relation

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda) &= \sum_{i=1}^N \left\{ \frac{\sigma(\eta)\sigma(u_N - \xi_i + \lambda_{21} + N\eta)}{\sigma(u_N - \xi_i + \eta)\sigma(\lambda_{21} + N\eta)} \prod_{j \neq i} \frac{\sigma(u_N - \xi_j)\sigma(\xi_i - \xi_j + \eta)}{\sigma(u_N - \xi_j + \eta)\sigma(\xi_i - \xi_j)} \right. \\ &\quad \left. \times Z_{N-1}(\{u_\alpha\}_{\alpha \neq N}; \{\xi_j\}_{j \neq i}; \lambda) \right\}. \end{aligned} \quad (5.5)$$

In the trigonometric limit, the recursive relation (5.5) recovers that of [6] for the six-vertex model.

## 5.2 The DW partition function

The recursive relation (5.4) and the property (4.1) uniquely determine  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$ , on the other hand the recursive relation (5.5) and the property (4.1) also fully fix the partition function. Using the Riemann identity (2.4) of the  $\sigma$ -function, one can check that the solution to each of recursive relations (5.4) and (5.5) gives rise to a symmetric function of  $\{u_\alpha\}$  as required. As a consequence, the two expressions of the partition function obtained by solving the recursive relations (5.4) and (5.5) respectively are equal since they are related to each other by some permutation of  $\{u_\alpha\}$ . Here, we present the result by resolving the recursive relation (5.5).

Using the property (4.1) and the recursive relation (5.5), we obtain the explicit expression of the partition function  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$

$$Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda) = \sum_{s \in S_N} \prod_{l=1}^N \left\{ \frac{\sigma(\eta)\sigma(u_l - \xi_{s(l)} + \lambda_{21} + l\eta)}{\sigma(u_l - \xi_{s(l)} + \eta)\sigma(\lambda_{21} + l\eta)} \right. \\ \left. \times \prod_{k=1}^{l-1} \frac{\sigma(u_l - \xi_{s(k)})\sigma(\xi_{s(l)} - \xi_{s(k)} + \eta)}{\sigma(u_l - \xi_{s(k)} + \eta)\sigma(\xi_{s(l)} - \xi_{s(k)})} \right\}, \quad (5.6)$$

where  $S_N$  is the permutation group of  $N$  indices. It is easy to see from the above explicit expression that  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$  is indeed a symmetry function of  $\{\xi_i\}$ .

## 6 Conclusions and discussions

We have introduced the DW boundary condition specified by the four boundary states (3.12)-(3.15) for the eight-vertex model on an  $N \times N$  square lattice. The boundary states are the two-parameter generalization of the all-spin-up and all-spin-down states and their dual states. With the DW boundary condition, we have obtained the properties (4.1), (4.2), (4.4), (4.5), (4.7) and (4.8) of the partition function  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$ . These properties enable us to derive the two recursive relations (5.4) and (5.5) of  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$ , whose trigonometric limits recover those corresponding to the six-vertex model. The recursive relation (5.4) or (5.5) together with (4.1) uniquely determines the partition function  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$ . Solving the recursive relations, we obtain the explicit expression (5.6) of the partition function.

Since the DW partition function (5.6) of the eight-vertex model reduces in the trigonometric limit to that of the six-vertex model which can be expressed in terms of a determinant [3], it is natural to think that (5.6) might be expressed in terms of a determinant. However,

it seems that the partition function (5.6) could only be expressed in terms of a sum of  $N$  determinants as follows:

$$Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda) = \frac{\prod_{\alpha=1}^N \prod_{j=1}^N \sigma(u_\alpha - \xi_j)}{\prod_{\alpha>\beta} \sigma(u_\alpha - u_\beta) \prod_{k<l} \sigma(\xi_k - \xi_l)} \frac{1}{\sigma(\sum_\alpha (u_\alpha - \xi_\alpha) + \lambda_{21} + N\eta)} \\ \times \sum_{k=1}^N \alpha_k \left\{ \frac{\sigma(\eta)}{\sigma(\lambda_{21} + N\eta - \gamma_k) \sigma(\lambda_{21} + \eta + \gamma_k)} \right\}^{N-1} \det \mathcal{M}(\gamma_k), \quad (6.1)$$

where  $\mathcal{M}(\gamma)$  is  $N \times N$  matrix with matrix elements are given by

$$\mathcal{M}(\gamma)_{\alpha j} = \frac{\sigma(u_\alpha - \xi_j + \lambda_{21} + \eta + \gamma) \sigma(u_\alpha - \xi_j + \lambda_{21} + N\eta - \gamma)}{\sigma(u_\alpha - \xi_j + \eta) \sigma(u_\alpha - \xi_j)}, \quad \alpha, j = 1, \dots, N. \quad (6.2)$$

The  $2N$  parameters  $\{\alpha_k; \gamma_k | k = 1, \dots, N\}$  in (6.1) satisfy the following equations

$$\sum_{i=1}^N \alpha_i \left\{ \frac{\sigma(\lambda_{21} + \gamma_i) \sigma(\lambda_{21} - \gamma_i + (N-1)\eta)}{\sigma(\lambda_{21} + \gamma_i + \eta) \sigma(\lambda_{21} - \gamma_i + N\eta)} \right\}^k \sigma(\lambda_{21} + \gamma_i + \eta) \sigma(\lambda_{21} - \gamma_i + N\eta) \\ = \frac{\sigma(\lambda_{21}) \sigma(\eta) \sigma(\lambda_{21} + (N-k)\eta)}{\sigma(\lambda_{21} + k\eta)}, \quad k = 0, \dots, N-1, \quad (6.3)$$

$$\sum_{i=1}^N \alpha_i \left\{ \frac{\sigma(\lambda_{21} + \gamma_i) \sigma(\lambda_{21} - \gamma_i + (N-1)\eta)}{\sigma(\lambda_{21} + \gamma_i + \eta) \sigma(\lambda_{21} - \gamma_i + N\eta)} \right\}^k \sigma(\gamma_i + (k+1-N)\eta) \sigma(k\eta - \gamma_i) \\ = 0, \quad k = 0, \dots, N-1. \quad (6.4)$$

The constraints (6.3) and (6.4) assure that the R.H.S. of (6.1) has the properties (4.1), (4.2), (4.4), (4.5) and (4.6) of the partition function  $Z_N(\{u_\alpha\}; \{\xi_j\}; \lambda)$ . Moreover, one can check that the R.H.S. of (6.1) also satisfies the relation (4.8). As is shown in section 5, these properties uniquely determine the partition function. Therefore, (6.1) gives an equivalent expression of the partition function (5.6).

## Acknowledgments

Financial support from the Australian Research Council is gratefully acknowledged.

*Note added:* After submitting our paper to the arXiv, we became aware that the partition function for the elliptic SOS model with the domain wall boundary condition has been obtained in [23, 24, 25] by different methods. Our approach is based entirely on the algebraic Bethe ansatz framework and can be used to obtain partition functions of the eight-vertex model with open boundary conditions [26]. We would like to thank V. Mangazeev for kindly drawing our attention to these references.

## Appendix A: The proof of (4.2)

In this appendix, we show that  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  is a symmetric function of  $\{u_\alpha\}$  and  $\{\xi_i\}$  separatively.

Regarding the tensor space indexed by  $\bar{1}, \dots, \bar{N}$  as the auxiliary space, (2.12) can be rewritten as

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) = & \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | \tilde{\phi}_{\lambda, \lambda - \eta \hat{1}}^{\bar{1}}(u_1) T_{\bar{1}}(u_1) \phi_{\lambda + \eta N \hat{2}, \lambda + \eta(N-1) \hat{2}}^{\bar{1}}(u_1) \dots \\ & \times \tilde{\phi}_{\lambda - \eta(N-1) \hat{1}, \lambda - \eta N \hat{1}}^{\bar{N}}(u_N) T_{\bar{N}}(u_N) \phi_{\lambda + \eta \hat{2}, \lambda}^{\bar{N}}(u_N) | \Omega^{(1)}(\lambda) \rangle. \end{aligned} \quad (\text{A.1})$$

Following [21, 22], let us introduce the face-type monodromy matrix with elements given by

$$T(m; m_0 | u)_\mu^j = \tilde{\phi}_{m + \eta \hat{j}, m}(u) T(u) \phi_{m_0 + \eta \hat{\mu}, m_0}(u), \quad j, \mu = 1, 2. \quad (\text{A.2})$$

Then the partition function  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  can be expressed in terms of the product of the matrix elements of the face-type monodromy matrix

$$\begin{aligned} Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) = & \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | T(\lambda - \eta \hat{1}; \lambda + \eta(N-1) \hat{2} | u_1)_2^1 T(\lambda - 2\eta \hat{1}; \lambda + \eta(N-2) \hat{2} | u_2)_2^1 \\ & \times \dots T(\lambda - \eta N \hat{1}; \lambda | u_N)_2^1 | \Omega^{(1)}(\lambda) \rangle. \end{aligned} \quad (\text{A.3})$$

The exchange relation (2.14) and the vertex-face correspondence relations (3.7) and (3.11) enable us to derive the following exchange relations for the operators (A.2)

$$\begin{aligned} & \sum_{i,j=1}^2 \bar{R}(u_1 - u_2; m)_{ij}^{kl} T(m + \eta \hat{j}; m_0 + \eta \hat{\nu} | u_1)_\mu^i T(m; m_0 | u_2)_\nu^j \\ & = \sum_{\alpha, \beta=1}^2 \bar{R}(u_1 - u_2; m_0)_{\mu \nu}^{\alpha \beta} T(m + \eta \hat{k}; m_0 + \eta \hat{\alpha} | u_2)_\beta^l T(m; m_0 | u_1)_\alpha^k. \end{aligned}$$

For the case of  $k = l = 1$  and  $\mu = \nu = 2$ , the above relations become

$$T(m + \eta \hat{1}; m_0 + \eta \hat{2} | u_1)_2^1 T(m; m_0 | u_2)_2^1 = T(m + \eta \hat{1}; m_0 + \eta \hat{2} | u_2)_2^1 T(m; m_0 | u_1)_2^1. \quad (\text{A.4})$$

This relation with  $T(m; m_0 | u)_2^1$  given by (A.2) implies that the partition function  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  (A.3) is a symmetric function of  $\{u_\alpha\}$ . Similarly, one can check that the partition function is also a symmetric function of  $\{\xi_i\}$ .

## Appendix B: The proofs of (4.7) and (4.8)

Firstly, let us prove the analytic property (4.7) of the partition function. Since the partition function is a symmetric function of  $\{u_\alpha\}$  and  $\{\xi_i\}$ , it is sufficient to prove (4.7) for the case of  $u_N = \xi_N - \eta$ . For this purpose, we need to rearrange the order of the product of R-matrices in the expression (2.11) of the partition function as follows

$$\begin{aligned}
Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) &= \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | \langle \bar{\Omega}^{(1)}(\lambda) | R_{\bar{1},N}(u_1 - \xi_N) R_{\bar{2},N}(u_2 - \xi_N) \dots R_{\bar{N},N}(u_N - \xi_N) \dots \\
&\quad \times R_{\bar{1},1}(u_1 - \xi_1) \dots R_{\bar{N},1}(u_N - \xi_1) | \bar{\Omega}^{(2)}(\lambda + \eta N \hat{2}) \rangle | \Omega^{(1)}(\lambda) \rangle. \\
&= \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | \tilde{\phi}_{\lambda, \lambda - \eta \hat{1}}^{\bar{1}}(u_1) \dots \tilde{\phi}_{\lambda - \eta(N-2)\hat{1}, \lambda - \eta(N-1)\hat{1}}^{\bar{N-1}}(u_{N-1}) \\
&\quad \times R_{\bar{1},N}(u_1 - \xi_N) \dots R_{\bar{N-1},N}(u_{N-1} - \xi_N) \\
&\quad \times \tilde{\phi}_{\lambda - \eta(N-1)\hat{1}, \lambda - \eta N \hat{1}}^{\bar{N}}(u_N) R_{\bar{N},N}(u_N - \xi_N) \phi_{\lambda - \eta(N-1)\hat{1}, \lambda - \eta N \hat{1}}^N(\xi_N) \\
&\quad \times R_{\bar{1},N-1}(u_1 - \xi_{N-1}) \dots R_{\bar{N},N-1}(u_N - \xi_{N-1}) \dots R_{\bar{N},1}(u_N - \xi_1) \\
&\quad \times \phi_{\lambda, \lambda - \eta \hat{1}}^1(\xi_1) \dots \phi_{\lambda - \eta(N-2)\hat{1}, \lambda - \eta(N-1)\hat{1}}^{N-1}(\xi_{N-1}) | \bar{\Omega}^{(2)}(\lambda + \eta N \hat{2}) \rangle. \tag{B.1}
\end{aligned}$$

The expressions (3.4)-(3.5) of the matrix elements of  $\bar{R}$  imply that

$$\text{Res}_{u=-\eta} R(u; m)_{11}^{11} = 0, \quad \text{Res}_{u=-\eta} R(u; m)_{21}^{12} = \frac{\sigma(\eta)\sigma(m_{21} - \eta)}{\sigma'(0)\sigma(m_{21})}. \tag{B.2}$$

Keeping the above equations in mind and using the vertex-face correspondence relation (3.10), we find

$$\begin{aligned}
&\text{Res}_{u_N = \xi_N - \eta} Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) \\
&= \frac{\sigma(\eta)\sigma(\lambda_{21} + (N-1)\eta)}{\sigma'(0)\sigma(\lambda_{21} + N\eta)} \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | \\
&\quad \times \tilde{\phi}_{\lambda, \lambda - \eta \hat{1}}^{\bar{1}}(u_1) \dots \tilde{\phi}_{\lambda - \eta(N-3)\hat{1}, \lambda - \eta(N-2)\hat{1}}^{\bar{N-2}}(u_{N-2}) \tilde{\phi}_{\lambda + \eta \hat{2} - \eta(N-1)\hat{1}, \lambda - \eta(N-1)\hat{1}}^{\bar{N}}(u_N) \\
&\quad \times R_{\bar{1},N}(u_1 - \xi_N) \dots R_{\bar{N-2},N}(u_{N-2} - \xi_N) \\
&\quad \times \tilde{\phi}_{\lambda - \eta(N-2)\hat{1}, \lambda - \eta(N-1)\hat{1}}^{\bar{N-1}}(u_{N-1}) R_{\bar{N-1},N}(u_{N-1} - \xi_N) \phi_{\lambda + \eta \hat{2} - \eta(N-1)\hat{1}, \lambda - \eta(N-1)\hat{1}}^N(\xi_N) \\
&\quad \times R_{\bar{1},N-1}(u_1 - \xi_{N-1}) \dots R_{\bar{N},N-1}(u_N - \xi_{N-1}) \dots R_{\bar{N},1}(u_N - \xi_1) \\
&\quad \times \phi_{\lambda, \lambda - \eta \hat{1}}^1(\xi_1) \dots \phi_{\lambda - \eta(N-2)\hat{1}, \lambda - \eta(N-1)\hat{1}}^{N-1}(\xi_{N-1}) | \bar{\Omega}^{(2)}(\lambda + \eta N \hat{2}) \rangle \\
&= \\
&\vdots \\
&= \frac{\sigma(\eta)\sigma(\lambda_{21} + (N-1)\eta)}{\sigma'(0)\sigma(\lambda_{21} + N\eta)} \prod_{l=1}^{N-1} \bar{R}(u_l - \xi_N; \lambda - \eta l \hat{1})_{12}^{12}
\end{aligned}$$

$$\begin{aligned}
& \times \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | \phi_{\lambda + \eta \hat{2}, \lambda}^N(\xi_N) \\
& \times \tilde{\phi}_{\lambda + \eta \hat{2}, \lambda + \eta \hat{2} - \eta \hat{1}}^1(u_1) \dots \tilde{\phi}_{\lambda + \eta \hat{2} - \eta(N-2) \hat{1}, \lambda + \eta \hat{2} - \eta(N-1) \hat{1}}^{N-1}(u_{N-1}) \\
& \times R_{\bar{1}, N-1}(u_1 - \xi_{N-1}) \dots R_{\bar{N-1}, N-1}(u_{N-1} - \xi_{N-1}) \\
& \times \tilde{\phi}_{\lambda + \eta \hat{2} - \eta(N-1) \hat{1}, \lambda - \eta(N-1) \hat{1}}^{\bar{N}}(u_N) R_{\bar{N}, N-1}(u_N - \xi_{N-1}) \phi_{\lambda - \eta(N-2) \hat{1}, \lambda - \eta(N-1) \hat{1}}^{N-1}(\xi_{N-1}) \\
& \times R_{\bar{1}, N-2}(u_1 - \xi_{N-2}) \dots R_{\bar{N}, N-2}(u_N - \xi_{N-2}) \dots R_{\bar{N}, 1}(u_N - \xi_1) \\
& \times \phi_{\lambda, \lambda - \eta \hat{1}}^1(\xi_1) \dots \phi_{\lambda - \eta(N-3) \hat{1}, \lambda - \eta(N-2) \hat{1}}^{N-2}(\xi_{N-2}) |\bar{\Omega}^{(2)}(\lambda + \eta N \hat{2})\rangle \\
& = \\
& \vdots \\
& = \frac{\sigma(\eta)\sigma(\lambda_{21} + (N-1)\eta)}{\sigma'(0)\sigma(\lambda_{21} + N\eta)} \prod_{l=1}^{N-1} \bar{R}(u_l - \xi_N; \lambda - \eta l \hat{1})_{12}^{12} \bar{R}(u_N - \xi_l; \lambda - \eta l \hat{1})_{21}^{21} \\
& \times \langle \Omega^{(2)}(\lambda + \eta N \hat{2}) | \phi_{\lambda + \eta \hat{2}, \lambda}^N(\xi_N) \\
& \times \tilde{\phi}_{\lambda + \eta \hat{2}, \lambda + \eta \hat{2} - \eta \hat{1}}^1(u_1) \dots \tilde{\phi}_{\lambda + \eta \hat{2} - \eta(N-2) \hat{1}, \lambda + \eta \hat{2} - \eta(N-1) \hat{1}}^{N-1}(u_{N-1}) \\
& \times R_{\bar{1}, N-1}(u_1 - \xi_{N-1}) \dots R_{\bar{N-1}, N-1}(u_{N-1} - \xi_{N-1}) \dots \\
& \times R_{\bar{1}, 1}(u_1 - \xi_1) \dots R_{\bar{N-1}, 1}(u_{N-1} - \xi_1) \\
& \times \phi_{\lambda + \eta \hat{2}, \lambda + \eta \hat{2} - \eta \hat{1}}^1(\xi_1) \dots \phi_{\lambda + \eta \hat{2} - \eta(N-2) \hat{1}, \lambda + \eta \hat{2} - \eta(N-1) \hat{1}}^{N-1}(\xi_{N-1}) \\
& \times \tilde{\phi}_{\lambda + \eta \hat{2}, \lambda}^{\bar{N}}(u_N) |\bar{\Omega}^{(2)}(\lambda + \eta N \hat{2})\rangle. \tag{B.3}
\end{aligned}$$

It is understood that  $u_N = \xi_N - \eta$  in the above equations. With help of the definitions (3.13) and (3.14) of the boundary states and the condition (3.8), we finally obtain the residue of  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  at the simple pole  $\xi_N - \eta$ :

$$\begin{aligned}
& \text{Res}_{u_N = \xi_N - \eta} Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) \\
& = \frac{\sigma(\eta)\sigma(\lambda_{21})}{\sigma'(0)\sigma(\lambda_{21} + \eta)} \prod_{l=1}^{N-1} \frac{\sigma(u_l - \xi_N)}{\sigma(u_l - \xi_N + \eta)} \prod_{j=1}^{N-1} \frac{\sigma(\xi_j - \xi_N + \eta)}{\sigma(\xi_j - \xi_N)} \\
& \times \tilde{\phi}_{\lambda + \eta \hat{2} + \eta(N-1) \hat{2}, \lambda + \eta \hat{2} + \eta(N-2) \hat{2}}^1(\xi_1) \dots \tilde{\phi}_{\lambda + \eta \hat{2} + \eta \hat{2}, \lambda + \eta \hat{2}}^{N-1}(\xi_{N-1}) \\
& \times \tilde{\phi}_{\lambda + \eta \hat{2}, \lambda + \eta \hat{2} - \eta \hat{1}}^1(u_1) \dots \tilde{\phi}_{\lambda + \eta \hat{2} - \eta(N-2) \hat{1}, \lambda + \eta \hat{2} - \eta(N-1) \hat{1}}^{N-1}(u_{N-1}) \\
& \times R_{\bar{1}, N-1}(u_1 - \xi_{N-1}) \dots R_{\bar{N-1}, N-1}(u_{N-1} - \xi_{N-1}) \dots \\
& \times R_{\bar{1}, 1}(u_1 - \xi_1) \dots R_{\bar{N-1}, 1}(u_{N-1} - \xi_1) \\
& \times \phi_{\lambda + \eta \hat{2} + \eta(N-1) \hat{2}, \lambda + \eta \hat{2} + \eta(N-2) \hat{2}}^1(u_1) \dots \phi_{\lambda + \eta \hat{2} + \eta \hat{2}, \lambda + \eta \hat{2}}^{N-1}(u_{N-1}) \\
& \times \phi_{\lambda + \eta \hat{2}, \lambda + \eta \hat{2} - \eta \hat{1}}^1(\xi_1) \dots \phi_{\lambda + \eta \hat{2} - \eta(N-2) \hat{1}, \lambda + \eta \hat{2} - \eta(N-1) \hat{1}}^{N-1}(\xi_{N-1})
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma(\eta)\sigma(\lambda_{21})}{\sigma'(0)\sigma(\lambda_{21} + \eta)} \prod_{l=1}^{N-1} \frac{\sigma(u_l - \xi_N)}{\sigma(u_l - \xi_N + \eta)} \prod_{j=1}^{N-1} \frac{\sigma(\xi_j - \xi_N + \eta)}{\sigma(\xi_j - \xi_N)} \\
&\quad \times Z_{N-1}(\{u_\alpha\}_{\alpha \neq N}; \{\xi_i\}_{i \neq N}; \lambda + \eta \hat{2}). \tag{B.4}
\end{aligned}$$

Therefore, we have completed the proof of (4.7).

Noting  $\bar{R}(u; m)_{ii}^{ii} = 1$  for any values of  $u, m$  and  $i = 1, 2$  and the initial condition (2.9) of  $R$ , by a similar procedure as above, one can show <sup>3</sup> that

$$Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda) |_{u_N = \xi_1} = Z_{N-1}(\{u_\alpha\}_{\alpha \neq N}; \{\xi_i\}_{i \neq 1}; \lambda). \tag{B.5}$$

The fact that  $Z_N(\{u_\alpha\}; \{\xi_i\}; \lambda)$  is a symmetric function of  $\{u_\alpha\}$  and  $\{\xi_i\}$  then leads to (4.8).

## References

- [1] V. E. Korepin, *Commun. Math. Phys.* **86**, (1982) 391.
- [2] A. G. Izergin, *Sov. Phys. Dokl.* **32** (1987), 878.
- [3] A. G. Izergin, D.A. Coker and V.E. Korepin, *J. Phys.* **A25** (1992), 4315.
- [4] F. H. L. Essler, H. Frahm, A. G. Izergin and V. E. Korepin, *Commun. Math. Phys.* **174** (1995), 191.
- [5] V. E. Korepin, N. M. Bololiubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1993.
- [6] N. Kitanine, J. M. Maillet and V. Terras, *Nucl. Phys.* **B 554** (1999), 647.
- [7] V. E. Korepin and P. Zinn-Justin, *J. Phys.* **A33**, (2000) 7053.
- [8] P.M. Bleher and V.V. Fokin, Exact solution of the six-vertex model with domain wall boundary conditions: Disordered phase, [math-ph/0510033](https://arxiv.org/abs/math-ph/0510033).
- [9] K. Sogo, *J. Phys. Soc. Jpn.* **62** (1993) 1887.
- [10] G. Kuperburg, *Int. Math. Res. Not.* **3** (1996) 139; *Ann. of Math.* **156** (2002), 835.
- [11] A. Caradoc, O. Foda and N. Kitanine, *J. Stat. Mech* (2006), P03012.

---

<sup>3</sup>In the proof of (B.5), it is convenient to keep the same order of the product of the R-matrices as that of (2.11) (c.f. (B.1)) in the calculation.

- [12] S.-Y. Zhao and Y.-Z. Zhang, *J. Math. Phys.* **48** (2007), 023504.
- [13] O. Foda, M. Wheeler and M. Zuparic, *J. Stat. Mech.* (2007), P10016; *J. Stat. Mech.* (2008), P02001.
- [14] R. J. Baxter, *Phys. Rev. Lett.* **26** (1971), 832; *Ann. Phys.* **76** (1973), 1; *Ann. Phys.* **76** (1973), 25; *Ann. Phys.* **76** (1973), 48.
- [15] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982.
- [16] E. T. Whittaker and G. N. Watson, *A course of modern analysis: 4th edn.*, Cambridge University Press, 2002.
- [17] M. Jimbo, T. Miwa and M. Okado, *Lett. Math. Phys.* **14** (1987), 123; *Nucl. Phys.* **B 300** (1988), 74.
- [18] W.-L. Yang and Y.-Z. Zhang, *Nucl. Phys.* **B 744** (2006), 312; *Nucl. Phys.* **B 789** (2008), 591.
- [19] A. Belavin and M. Jimbo, *Int. J. Mod. Phys.* **A 19** (2004), 50.
- [20] W.-L. Yang, A. Belavin and R. Sasaki, *Nucl. Phys.* **B 710** (2005), 614.
- [21] W.-L. Yang and R. Sasaki, *Nucl. Phys.* **B 679** (2004), 495.
- [22] W.-L. Yang, R. Sasaki and Y.-Z. Zhang, *JHEP* **0409** (2004), 046.
- [23] H. Rosengren, An Izergin-Korepin-type identity for the 8VSOS model, with applications to alternating sign matrices, [arXiv:0801.1229](https://arxiv.org/abs/0801.1229).
- [24] S. Pakuliak, V. Rubtsov and A. Silantyev, *J. Phys.* **41** (2008), 295204.
- [25] A. V. Razumov and Yu. G. Stroganov, Three-coloring statistical model with domain wall boundary conditions. I. Functional equations, [arXiv:0805.0669](https://arxiv.org/abs/0805.0669); Three-coloring statistical model with domain wall boundary conditions. II. Trigonometric limit, [arXiv:0812.2654](https://arxiv.org/abs/0812.2654).
- [26] W.-L. Yang and Y.-Z. Zhang, in preparation.